

# Maximal Independent Sets in Generalised Caterpillar Graphs

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## Abstract

A caterpillar graph is a tree which on removal of all its pendant vertices leaves a chordless path. The chordless path is called the backbone of the graph. The edges from the backbone to the pendant vertices are called the hairs of the caterpillar graph. Ortiz and Villanueva (C.Ortiz and M.Villanueva, Discrete Applied Mathematics, 160(3): 259-266, 2012) describe an algorithm, linear in the size of the output, for finding a family of maximal independent sets in a caterpillar graph.

In this paper, we propose an algorithm, again linear in the output size, for a generalised caterpillar graph, where at each vertex of the backbone, there can be any number of hairs of length one and at most one hair of length two.

**Keywords:** Maximal Independent Set; MIS; Caterpillar Graphs; Generalised Caterpillar Graphs; Generating all MIS; Algorithm

## 1 Introduction

A caterpillar graph  $C(P_k)$  (See Figure 1) is a tree which on removal of all its pendant vertices (vertices  $h_i$  and  $l_j$  in the figure) results in a chordless path  $P_k = \{v_1, v_2, \dots, v_k\}$  of  $k$  vertices. The path  $P_k$  is called the backbone of the caterpillar graph  $C(P_k)$  and the edges from the backbone to the pendant vertices (edges  $(v_i, h_i)$  and  $(v_j, l_j)$  in Figure 1) are called its hairs. In  $C(P_k)$  all hairs are of length one. Harary and Schwenk [3], introduced Caterpillar graphs by saying:

“Caterpillar is a tree which metamorphoses into a path when its cocoon of endpoints is removed”.

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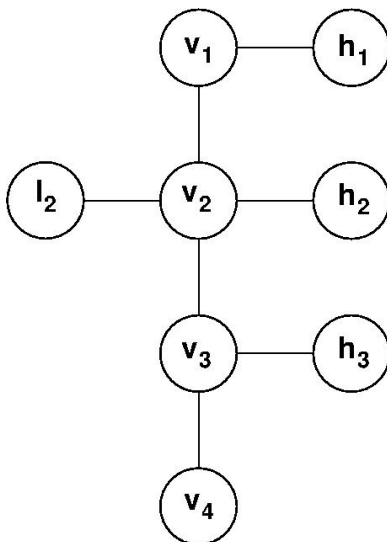


Figure 1: An example for a caterpillar graph

In chemical graph theory, caterpillar graphs are useful in studying topological properties of benzenoid hydrocarbons[2]. In fact, Basil and Sherif [2] observe that

“It is amazing that nearly all graphs that played an important role in what is now called “chemical graph theory” may be related to caterpillar trees.”

Ortiz and Villanueva [4] describe an algorithm for enumerating a family of maximal independent sets in caterpillar graphs. The algorithm takes time linear in the size of the output, i.e., is linear in the sum of sizes of all maximal independent sets. They also propose  $CC^2(P_k)$ , a generalisation in which hairs have length exactly two.

In this paper, we consider a still more generalised version of caterpillar graphs (See Figure 2). In this generalised version, we allow a backbone vertex  $v_i$  to have up to one hair of length two and any number of hairs of length one; in particular  $v_i$  may have no hairs at all. If  $P_k = \{v_1, \dots, v_k\}$ , a chordless path of length  $k$  is the backbone, then we denote the generalised caterpillar graph by  $C^{1,2}(P_k)$ .

A complete caterpillar graph  $CC(P_k)$  is a (usual) caterpillar graph such that there is at least one hair at each of its backbone vertices. The graph of Figure 1 is not complete as vertex  $v_4$  does not have any hair. The contraction graph  $G_k$  of a (usual) caterpillar graph  $C(P_k)$  is the graph obtained by contracting, for each backbone vertex  $v_i$  of the  $C(P_k)$ , all the pendant vertices incident at  $v_i$  to a single vertex [4].

An independent set or a stable set is a set of vertices in a graph such that no

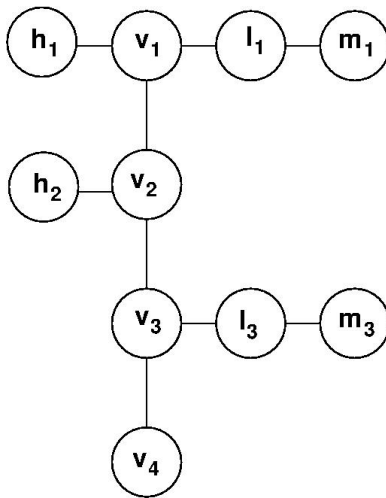


Figure 2: An example for a generalised caterpillar graph

two vertices in the set are adjacent. That is, it is a set  $I$  of vertices of a graph  $G$  such that if  $I$  contains two vertices, say  $a$  and  $b$ , then  $ab$  is not an edge of  $G$ . The size or cardinality of an independent set  $I$  is the number of vertices in the set  $I$ . An independent set  $I$  will be called a maximal independent set if every vertex  $v$  is either in  $I$  or is adjacent to a vertex in  $I$ .

Valiant [6] shows that the problem of counting the number of maximal independent sets is  $\#P$ -complete for general graphs. Tsukiyama et al. [5] show that we can enumerate a family of maximal independent sets of a general connected graph in  $O(nmm(G))$  time; here  $n$  is the number of vertices,  $m$  is the number of edges and  $m(G)$  is the number of maximal independent sets of a graph  $G$ . Ortiz and Villanueva [4] show that  $m(C(P_k))$ , the number of maximal independent sets of a (usual) caterpillar graph  $C(P_k)$  is the same as  $m(G_k)$ , the number of maximal independent sets of its contraction graph,  $G_k$ . They give an algorithm to find a family of maximal independent sets of a caterpillar graph in time polynomial in the number of maximal independent sets. In this paper, we obtain a similar result for generalised caterpillar graphs.

Vertices of generalised caterpillar graph can be partitioned into stages. Vertex  $v_i$  together with vertices in hairs incident at  $v_i$  will be said to be at stage  $i$ . Formally, if we delete vertex  $v_i$  from  $C^{1,2}(P_k)$ , the graph may split into several components. The vertices in components which do not contain either vertex  $v_{i-1}$  or  $v_{i+1}$  will be at stage  $i$  (along with vertex  $v_i$ ).

Let  $x_1, x_2, \dots, x_{m_i}$  be the pendant vertices in hairs of length one at stage  $i$  of  $C^{1,2}(P_k)$ . If any of these vertices is in a maximal independent set  $S$ , then  $v_i$  cannot be in the independent set  $S$ . Conversely, if  $v_i$  is not in  $S$ , then we have to put all these vertices in the maximal independent set  $S$ . Thus, any pendant vertex belonging to a hair of length one at stage  $i$  is contained in a maximal

independent set  $S$ , if and only if all pendant vertices at stage  $i$  are contained in  $S$ .

Hence, the number of maximal independent sets of  $C^{1,2}(P_k)$  is independent of the number of hairs of length one at each stage (provided there is at least one such hair). Hence from now on, we assume that  $C^{1,2}(P_k)$  has at most one hair of length one at each stage  $i$  (wherever there is at least one such hair) and we denote this hair by  $v_i h_i$ . We will denote the hair of length two at stage  $i$  by  $v_i l_i m_i$ , wherever it exists.

## 2 Structure of Maximal Independent Sets in generalised Caterpillar Graphs

If  $S$  is any maximal independent set of  $C^{1,2}(P_k)$ , let  $S_i$  be the subset of  $S$  containing only vertices at stage  $i$ . Clearly,  $S = S_1 \cup S_2 \cup \dots \cup S_k$ . In a caterpillar graph, the only vertices from two different stages, adjacent to each other are the vertices  $v_i$  and  $v_{i+1}$  (or  $v_{i-1}$  and  $v_i$ ).

In case, if a hair of length one is present at stage  $i$  (hair of length two may or may not be there), then either  $h_i$  or  $v_i$  (but not both) will be present in any maximal independent set. Hence, in this case, exactly one of  $h_i$  or  $v_i$  will be in the set  $S_i$ .

Similarly, if a hair of length two is present at stage  $i$  (hair of length one may or may not be present), then either  $m_i$  or  $l_i$  (but not both) will be present in any maximal independent set. Hence, in this case, exactly one of  $m_i$  or  $l_i$  will be present in  $S_i$ .

Also observe that as  $v_i$  and  $l_i$  are adjacent, both of them cannot be present in any independent set.

In general, either both hairs of length one and two, or only one or neither may be present at stage  $i$ . In all there are exactly three possibilities.

**Case 1** ( $v_i \in S_i$ ): If  $v_i \in S_i$ , then if a hair of length two is present then  $l_i \notin S_i$  and hence  $m_i \in S_i$ . Moreover, even if we have hair of length one, then as  $v_i$  and  $h_i$  are adjacent,  $h_i \notin S_i$ , thus

$S_i = \{v_i, m_i\}$  if hair of length two is present  
 $S_i = \{v_i\}$  otherwise

**Case 2** ( $l_i \in S_i$ ): If  $l_i \in S_i$ , then  $v_i \notin S_i$  and  $m_i \notin S_i$ . If a hair of length one is present at stage  $i$ , then as  $v_i \notin S_i$ ,  $h_i \in S_i$ , thus, in this case

$S_i = \{h_i, l_i\}$  if hair of length one is present  
 $S_i = \{l_i\}$  otherwise

**Case 3** ( $v_i, l_i \notin S_i$ ) If length one hair is present, then as  $v_i \notin S_i$  we must have  $h_i \in S_i$ . If length two hair is present then as  $l_i \notin S_i$ ,  $m_i \in S_i$ . Hence, in this case,

$S_i = \{h_i, m_i\}$  if hairs of length one and two are both present  
 $S_i = \{h_i\}$  if only hair of length one is present  
 $S_i = \{m_i\}$  if only hair of length two is present  
 $S_i = \Phi$  otherwise

Thus, in each of these cases, all vertices at stage  $i$ , except possibly  $v_i$ , will either be in the independent set  $S_i$  or will be adjacent to a vertex in  $S_i$ . If  $v_i \in S_i$ , the set  $S_i$  will be a maximal independent set of the subgraph at stage  $i$ .

Thus, we have the following lemma:

**Lemma 1:**  $S = S_1 \cup S_2 \cup \dots S_k$  is a Maximal Independent Set of generalised caterpillar graph  $C^{1,2}(P_k)$  if and only if following conditions hold:

- (1) for each  $v_i$  which is not adjacent to a vertex in  $S_i$ , either  $v_{i-1} \in S_{i-1}$  or  $v_{i+1} \in S_{i+1}$ .
- (2) both  $v_{i-1} \in S_{i-1}$  and  $v_i \in S_i$  should not simultaneously hold.

**Proof:** As each  $S_i$  is an independent set, and as each vertex of  $S_i$  except possibly  $v_i$  is either in  $S_i$ , or has a neighbour in  $S_i$ , the set  $S$  will be a maximal independent set iff either each  $v_i \in S$  or each  $v_i$  has a neighbour in  $S$ . Vertex  $v_i \in S$  iff  $v_i \in S_i$ .

If  $v_i \notin S_i$ , then  $v_i$  has a neighbour in  $S$  iff one of the following conditions hold

- (1) either  $v_i$  has a neighbour in  $S_i$ , or
- (2)  $v_{i-1} \in S$  or
- (3)  $v_{i+1} \in S$ .

For the set  $S$  to be independent, clearly the second condition must hold. The lemma thus follows.  $\square$

### 3 Finding Maximal Independent Sets in generalised Caterpillar Graphs

Let us assume that  $S = S_1 \cup S_2 \cup \dots S_k$  is a maximal independent set of  $C^{1,2}(P_k)$ . Then, for  $i \leq k$ , we classify the set  $S_i$  depending upon the “status” of vertex  $v_i$ .

**Type 1:** If  $v_i \in S_i$ , then we will say  $S_i$  is of Type 1.

**Type 2:** If  $v_i \notin S_i$ , but some neighbour of  $v_i$  is in  $S_i$ , then we will say  $S_i$  is of Type 2. In this case either  $h_i \in S_i$  or  $l_i \in S_i$ .

**Type 3:** If no neighbour of  $v_i$  is in  $S_i$ , but  $v_{i-1} \in S_{i-1}$ , then we will say  $S_i$  is of Type 3.

**Type 4:** If  $v_{i-1} \notin S_{i-1}$ ,  $v_i \notin S_i$  and no neighbour of  $v_i$  is in  $S_i$ , then we will say that  $S_i$  is of Type 4. In this case, for the set  $S$  to be maximal,  $v_{i+1} \in S_{i+1}$ .

The last stage  $S_k$  cannot be of Type 4, as in that case neither  $v_k \in S$  nor  $v_{k-1} \in S$ . And hence, as  $v_k$  does not have a neighbour in  $S_k$ ,  $v_k$  will not have any neighbour in  $S$  (there is no vertex  $v_{k+1}$ ), violating maximality. Further, as  $S_1$  is the first independent set,  $S_1$  cannot be of Type 3 (there is no vertex  $v_0$ ).

We store the types of hairs present at stage  $i$  in an array  $T$ . The entry  
 $T[i] = 0$  if there are no hairs at stage  $i$   
 $T[i] = 1$  if there are only length one hairs at stage  $i$   
 $T[i] = 2$  if there is only length two hair at stage  $i$   
 $T[i] = 3$  if stage  $i$  has both types of hairs

Then the table below summarises the discussion above and gives the list of all  $S_i$ 's of each type.

$T[i]$	$S_i$ of type 1	$S_i$ of type 2	$S_i$ of type 3	$S_i$ of type 4
0	$\{v_i\}$	none	$\phi$	$\phi$
1	$\{v_i\}$	$\{h_i\}$	none	none
2	$\{v_i, m_i\}$	$\{l_i\}$	$\{m_i\}$	$\{m_i\}$
3	$\{v_i, m_i\}$	$\{h_i, l_i\}, \{h_i, m_i\}$	none	none

Table 1: The possible instances of  $S_i$  and their types depending on the hairs present

**Theorem 1:**  $S = S_1 \cup S_2 \cup \dots \cup S_k$  is a maximal independent set of a generalised caterpillar graph  $C^{1,2}(P_k)$  if and only if for  $1 \leq i \leq k-1$ ,  $(S_i, S_{i+1})$  is of one of the following forms, and  $S_k$  is not of the type 4.

- (1) (type 1, type 2)
- (2) (type 1, type 3)
- (3) (type 2, type  $x$ ), where  $x$  is 1, 2 or 4
- (4) (type 3, type  $x$ ), where  $x$  is 1, 2 or 4
- (5) (type 4, type 1)

**Proof:** Let us prove the ‘only if’ part first. Let us assume that  $S$  is a maximal independent set of  $C^{1,2}(P_k)$ . We have one of the following three cases, depending on the type of  $S_i$ .

- (1) If  $S_i$  is of Type 1, then as  $v_i \in S_i$ ,  $S_{i+1}$  cannot be of Type 4. As  $v_i \in S_i$ ,  $v_{i+1} \notin S_{i+1}$ , hence  $S_{i+1}$  cannot be of Type 1. If  $v_{i+1}$  has a neighbour in  $S_{i+1}$ , then  $S_{i+1}$  will be of Type 2, otherwise of Type 3.
- (2) If  $S_i$  is of Type 4, then as we saw earlier,  $v_{i+1} \in S_{i+1}$ , and hence  $S_{i+1}$  will be of Type 1.
- (3) If  $S_i$  is of Type 2 or of Type 3, then as  $v_i \notin S_i$ ,  $S_{i+1}$  cannot be of Type 3. If  $v_{i+1} \in S_{i+1}$ , then  $S_{i+1}$  will be of Type 1. If a neighbour of  $v_{i+1}$  is in  $S_{i+1}$ , then  $S_{i+1}$  will be of Type 2. If  $v_{i+1} \notin S_{i+1}$ , and if it does not have any neighbour in  $S_{i+1}$ , then  $S_{i+1}$  will be of Type 4.

To prove the ‘if’ part, we need to prove that for  $1 \leq i \leq k$ , if  $(S_i, S_{i+1})$  is in one of the forms listed (and  $S_k$  is not of Type 4), then  $S$  is a maximal independent set. As each vertex at stage  $i$ , except possibly  $v_i$ , is either in  $S_i$  or is adjacent to a vertex in  $S_i$ , we only need to show that

- (a) Both  $v_i \in S_i$  and  $v_{i+1} \in S_{i+1}$  cannot simultaneously hold.
- (b) If  $v_i \notin S_i$ , then  $v_i$  has a neighbour in  $S$ .

If  $v_i \in S_i$ , then  $S_i$  will be of Type 1, and if  $v_{i+1} \in S_{i+1}$ , then  $S_{i+1}$  will also be of Type 1. As we do not have the form (Type 1, Type 1), the first condition

holds.

If  $v_i \notin S_i$ , then  $S_i$  will not be of Type 1. If  $S_i$  is of Type 2, then  $v_i$  will have a neighbour in  $S_i$ , and hence in  $S$ . If  $S_i$  is of Type 3, then as the only permissible form for  $(S_{i-1}, S_i)$  is (Type 1, Type 3). Thus,  $S_{i-1}$  has to be of Type 1, and  $v_{i-1} \in S_{i-1}$ ; or  $v_i$  will have a neighbour  $v_{i-1}$  in  $S_{i-1}$ , and hence in  $S$ .

Finally, if  $S_i$  is of Type 4, then as the only permissible form is (Type 4, Type 1). Thus,  $S_{i+1}$  has to be of Type 1 and  $v_{i+1} \in S_{i+1}$  (for  $i \neq k$ ) and so  $v_{i+1}$  will have a neighbour in  $S_{i+1}$ , or in  $S$ . Hence,  $S$  will be both independent and maximal.  $\square$

## 4 Finding Family of Maximal Independent Sets

From Theorem 1, if  $(S_i, S_{i+1})$  is of one of the forms listed, then  $S = S_1 \cup S_2 \cup \dots \cup S_k$  will be a maximal independent set. Thus, to find a family of all maximal independent sets, we need to find all such valid sequences. For this, we construct a directed  $k$ -level graph  $L_k$  such that any maximal independent set in the generalised caterpillar graph corresponds to a source-sink (source to sink) path in  $L_k$ .

A  $k$ -level graph  $G = (V, E, \phi)$  with  $k \leq n$  is a graph with an assignment of levels  $\phi : V \rightarrow \{1, 2, \dots, k\}$  that partitions the vertex set into  $k$  pairwise disjoint subsets,  $V_1, V_2, \dots, V_k$  such that  $V = V_1 \cup V_2 \cup \dots \cup V_k$ . Further, if  $(uv)$  is an edge in  $G$ , then  $u$  and  $v$  are not in the same level [1]. In our level graph  $L_k$ , the edges are only from level  $i$  to level  $i + 1$ .

We will denote the set of vertices of  $L_k$  by  $U$ . The  $k$  levels in  $L_k$  are numbered from 1 to  $k$ ; level  $i$  in  $L_k$  corresponding to stage  $i$  in  $C^{1,2}(P_k)$ . Roughly speaking, at each level  $i$ , the vertices in  $L_k$  will correspond to one possible instance of  $S_i$ . The total number of vertices present at any level  $i$  of the level graph, will depend on the types of hairs present at stage  $i$ . We will see that the number of vertices at each level will be at most five.

We will use two labels “*type*” and “*index*” on vertices of  $L_k$ . Type of a (new) vertex will correspond to the type of corresponding  $S_i$ . Index will be one in all but one case. From Table 1, observe that, in all but one case, for any value of  $T[i]$ , there is at most one instance of  $S_i$ . If  $T[i] = 3$ , then there are two instances of  $S_i$ . We use the label “*index*” to distinguish these cases.

In more detail, we add vertices at level  $i$  as follows. First, for each  $i$ , we first add a vertex  $p_i$  and set  $type(p_i) = 1$  and  $index(p_i) = 1$ . This will correspond to the case when  $v_i \in S_i$ .

Depending upon type of hairs present at  $v_i$ , we add other vertices at level  $i$  as follows:

**Only Length one hairs ( $T[i] == 1$ ):** Add a vertex  $s_i$  and set  $type(s_i) = 2$  and  $index(s_i) = 1$ . This will correspond to the case when  $S_i$  is of type 2, and  $S_i = \{h_i\}$ .

**Only Length two hairs ( $T[i] == 2$ ):** In this case, also, we first add a vertex  $s_i$  and set  $type(s_i) = 2$  and  $index(s_i) = 1$ . This will correspond to the case when  $S_i$  is of type 2, and  $S_i = \{l_i\}$ . Further, we also

- (a) for  $i \geq 2$ , we add a vertex  $t_i$  and set  $type(t_i) = 3$  and  $index(t_i) = 1$ .  
(b) and for  $i \leq k-1$ , we add a vertex  $u_i$  and set  $type(u_i) = 4$  and  $index(u_i) = 1$ .

These cases correspond to the case when  $S_i$  is of type 3 or of type 4. In these cases,  $S_i = \{m_i\}$ .

**Both Length one and two hairs** ( $T[i] == 3$ ): Add two vertices  $q_i$  and  $r_i$  and set  $type(q_i) = type(r_i) = 2$ ;  $index(q_i) = 1$  and  $index(r_i) = 2$ .

This corresponds to the case when  $S_i$  is of type 2. Here  $S_i$  can be either  $\{h_i, l_i\}$  or  $\{h_i, m_i\}$ , thus we need two vertices, one for each case. We also use the label “*index*” to distinguish these two cases.

**No Hairs** ( $T[i] == 0$ ): In this case

- (a) for  $i \geq 2$ , we add a vertex  $t_i$  and set  $type(t_i) = 3$  and  $index(t_i) = 1$ .  
(b) and for  $i \leq k-1$ , we add a vertex  $u_i$  and set  $type(u_i) = 4$  and  $index(u_i) = 1$ .

These cases correspond to the case when  $S_i$  is of type 3 or of type 4. In these cases,  $S_i = \Phi$ .

This is summarised in Table 2 below:

vertex	type	index	Value of $T[i]$	set $S_i$
$p_i$	1	1	0, 1, 2 or 3	$v_i \in S_i$
$s_i$	2	1	1 or 2	$S_i = \{h_i\}$ or $S_i = \{l_i\}$
$q_i$	2	1	3	$S_i = \{h_i, l_i\}$
$r_i$	2	2	3	$S_i = \{h_i, m_i\}$
$t_i$	3	1	0 or 2	$S_i = \{m_i\}$ or $S_i = \Phi$
$u_i$	4	1	0 or 2	$S_i = \{m_i\}$ or $S_i = \Phi$

Table 2: The vertices at level  $i$  in  $L_k$  and the corresponding  $S_i$

Next we add following edges. We add an edge from a vertex  $a$  in level  $i$  to a vertex  $b$  in the next level  $i+1$ , if  $(type(a), type(b))$  is one of the following (listed in Theorem 1):

(1, 2), (1, 3), (2, 1), (2, 2), (2, 4), (3, 1), (3, 2), (3, 4) or (4, 1)

Formally, we add the following edges from a vertex  $a$  of level  $i$  to a vertex  $b$  of level  $i+1$ , for  $1 \leq i \leq k-1$ :

- (a) For each vertex  $a$  of type 1, we add an edge  $(a, b)$  in  $L_k$ , if vertex  $b$  is either of type 2 or of type 3.  
(b) For each vertex  $a$  of type 2 or type 3, we add an edge  $(a, b)$ , if vertex  $b$  is of type 1, type 2, or of type 4 (i.e.,  $b$  is not of type 3).  
(c) For each vertex  $a$  of type 4 we add an edge  $(a, b)$ , if vertex  $b$  is of type 1.

All vertices at Level 1 will be treated as sources and all vertices at level  $k$  as sinks, and hence any path from level 1 to level  $k$  in  $L_k$  will be a source-sink path.

As these edges correspond exactly to valid  $(S_i, S_{i+1})$  pairs of Theorem 1, hence again from Theorem 1, it follows that any source-sink path in  $L_k$  will correspond to a valid set of sequence  $S_1, S_2, \dots, S_k$  and conversely.



Thus, if we find all source-sink paths in  $L_k$ , we can find all the maximal independent sets in  $C^{1,2}(P_k)$ . This can be done by using a generalised depth first procedure (DFS), similar to the one used by [4].

We basically, put the start vertex in a stack and then for each neighbour of the start vertex (as the new start vertex), we call the procedure recursively. We stop when the start vertex for the current call is a sink vertex, and print the entire stack; and also remove this vertex from the stack (backtrack to the previous level).

The number of source-sink paths in  $L_k$  will be the number of maximal independent sets in  $C^{1,2}(P_k)$ . We can easily obtain maximal independent sets from the source-sink path. Let  $w_1, w_2, \dots, w_k$  be the vertices in a source sink path of  $L_k$ . Then we reconstruct the  $S_i$  corresponding to each  $P_i$  as follows (see Table 1):

If  $type(w_i) == 1$ , then  
if  $T[i] == 3$  or  $T[i] == 2$  then  $S_i = \{v_i, m_i\}$   
else  $S_i = \{v_i\}$

If  $type(w_i) == 3$  or  $type(w_i) == 4$ , then  
if  $T[i] == 2$  then  $S_i = \{m_i\}$   
if  $T[i] == 0$  then  $S_i = \Phi$

If  $type(w_i) == 2$ , then  
if  $(T[i] == 1)$  then  $S_i = \{h_i\}$   
if  $(T[i] == 2)$  then  $S_i = \{l_i\}$   
if  $T[i] = 3$   
    if  $index(w_i) == 1$  then  $S_i = \{h_i, l_i\}$ ,  
    else if  $index(w_i) == 2$  then  $S_i = \{h_i, m_i\}$

A theorem similar to that of Ortiz and Villanueva [4], also holds for the generalised caterpillar graph.

**Theorem 2:** We can enumerate all maximal independent sets of  $C^{1,2}(P_k)$  in  $O(km(C^{1,2}(P_k)))$  time, where  $m(C^{1,2}(P_k))$  is the number of maximal independent sets of  $C^{1,2}(P_k)$ .

**Proof:** For each source-sink path  $P$  of  $L_k$ , generalised depth first procedure is called once for each vertex of  $P$ . Since any edge in  $L_k$  is between adjacent levels, the number of vertices in the path is  $k + 1$ . Hence there are only  $O(k)$  calls to the procedure. As each call takes  $O(1)$  time, except when we reached a sink, in which case it takes  $O(k)$  time. But as we reach sink only once for a path, the algorithm takes  $O(k)$  time, for each path found.

As there are  $m(C^{1,2}(P_k))$  such paths, the algorithm takes  $O(km(C^{1,2}(P_k)))$  time.  $\square$

As each maximal independent set has  $k$  vertices, the algorithm is essentially linear in the output size.

## 5 Example

Let us illustrate the algorithm for the graph given in Figure 2, we have the following vertices of type 1 in it:

Level 1:  $p_1$ ,  $type(p_1) = 1$ ,  $index(p_1) = 1$

Level 2:  $p_2$ ,  $type(p_2) = 1$ ,  $index(p_2) = 1$

Level 3:  $p_3$ ,  $type(p_3) = 1$ ,  $index(p_3) = 1$

Level 4:  $p_4$ ,  $type(p_4) = 1$ ,  $index(p_4) = 1$

We have the following vertices of type 2:

Level 1:  $T[i] = 3$ . Two vertices  $q_1$  and  $r_1$ ,  $type(q_1) = 2$ ,  $index(q_1) = 1$ ,  $type(r_1) = 2$ ,  $index(r_1) = 2$

Level 2:  $T[i] = 1$ . Vertex  $s_2$ ,  $type(s_2) = 2$ ,  $index(s_2) = 1$

Level 3:  $T[i] = 2$ . Vertex  $s_3$ ,  $type(s_3) = 2$ ,  $index(s_3) = 1$

Level 4:  $T[i] = 0$ . No vertex.

We have the following vertices of type 3 in it:

Level 1:  $T[i] = 3$ . None

Level 2:  $T[i] = 1$ . None

Level 3:  $T[i] = 2$ . Vertex  $t_3$ ,  $type(t_3) = 3$ ,  $index(t_3) = 1$

Level 4:  $T[i] = 0$ . Vertex  $t_4$ ,  $type(t_4) = 3$ ,  $index(t_4) = 1$

The following are vertices of type 4.

Level 1:  $T[i] = 3$ . None

Level 2:  $T[i] = 1$ . None

Level 3:  $T[i] = 2$ . Vertex  $u_3$ ,  $type(u_3) = 4$ ,  $index(u_3) = 1$

Level 4:  $T[i] = 0$ . None as this is the last level

The edges in the example are as shown in Figure 3.

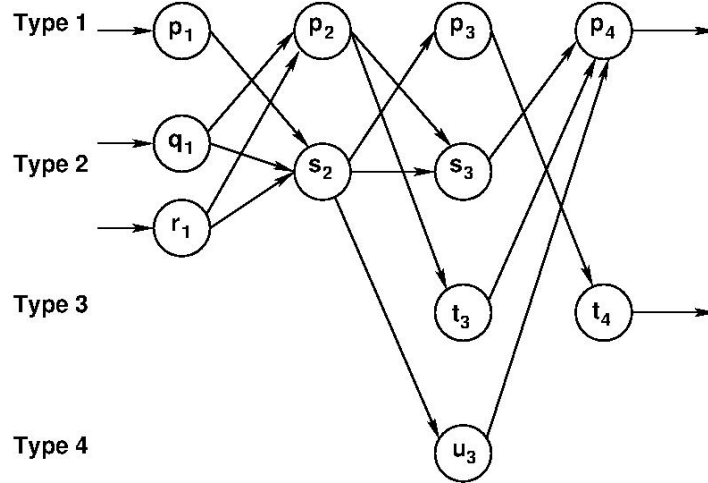


Figure 3: Construction of  $L_k$  from  $C^{1,2}(P_k)$

Each source-sink path corresponds to a maximal independent set. For example, the path  $p_1 - s_2 - u_3 - p_4$  in  $L_k$  corresponds to the maximal independent

set  $\{v_1, m_1, h_2, m_3, v_4\}$  in  $C^{1,2}(P_k)$ . Similarly, the path  $r_1 - s_2 - p_3 - t_4$  in  $L_k$  corresponds to the maximal independent set  $\{h_1, m_1, h_2, v_3, m_3\}$  in  $C^{1,2}(P_k)$ .

## Conclusions

We discuss the problem of finding a family of maximal independent sets in a generalised caterpillar graph. We show that this problem can also be reduced to the problem of finding all source-sink paths in a level graph. The proposed algorithm takes time linear in the output size (total number of vertices in all maximal independent sets).

Further, we believe, that this algorithm can be extended for another generalisation of caterpillar graph, where each vertex of backbone has bounded number of hairs of length more than one. It may also be possible to generalise the algorithm for some other generalisations of caterpillar graphs, possibly having a different set of hairs. We may have to identify the new set of possible  $S_i$ 's at each stage  $i$  and classifying them into possibly some other different "types".

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